Research Statement

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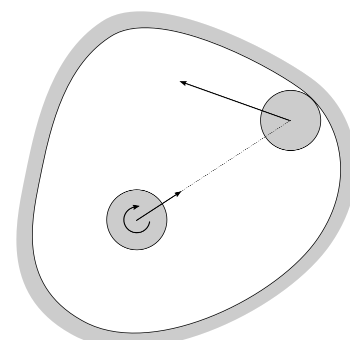
My research is on physically motivated billiard type dynamical systems, continuing the work I began with my dissertation on no-slip billiards. I am interested in exploring what can be proved analytically as well as what can be discovered numerically. Involving students in my research is a priority, and in particular numerical explorations have proven to be an entry point for students who do not necessarily have an advanced analytic background.

1 Introduction

Mathematical billiards, based on the simple "angle in equals angle out" rule, were formalized by Birkhoff nearly a hundred years ago, and used in the models of Galton, Lorentz, and others even earlier. Nonetheless, fundamental questions remain open and continue to inspire study, and new applications are found every year in areas as diverse as microorganism locomotion, robot motion planning, and cell phone tower distributions. Many natural alternatives to the standard model, for example no-slip billiards which incorporate rotation of the particles, are yet to be understood. My research aims to understand the dynamics in these more general settings, filling in a broad picture by using the following framework.



In no-slip billiards linear and angular momentum may be exchanged at collisions.

- Begin with a physically motivated particle or collision model.
- Formulate a detailed differential geometric framework.
- Explore the dynamical system given by the model numerically using computational tools.
- Where possible, establish the dynamics formally with analytic results.
- Use the tools of statistical mechanics to understand aspects of the macroscopic behavior and apply the model.

1.1 Current Areas of Focus

Currently, I am pursuing the outlined program as manifested in several areas:

1. **No-slip Billiards:** This model, in which particles rotate and may conservatively exchange linear and angular momentum at collisions, arises as a natural alternative on equal footing with the standard specular collisions.

2. **Nonholonomic Rolling Systems:** Particles rolling without slipping have interesting connections to no-slip billiards and are also of interest in their own right as a nonholonomic system.

3. **Lensed Billiards:** This model, in which billiards may refract or reflect according to potentials assigned to regions of the table, is a natural generalization of standard billiards considered from an optical perspective.

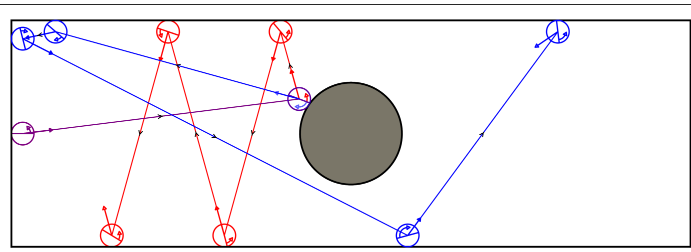
4. **Machine Learning of Billiard Dynamical Systems:** The structure and versatility of billiard dynamical systems makes them suitable for experiments in machine learning. We have been able to use orbits of no-slip billiards as input data for neural networks which successfully learn the phase space and predict orbits.

Section 2 gives a brief introduction to some of the systems, as well as information about collaborators, completed work and open problems, and references for further reading. Section 3 gives a summary of past and possible future projects ideal for undergraduate research.

2 An introduction to alternative billiard models

2.1 No-slip billiards

The idea of no-slip or "rough" collisions has been considered at least since Richard Garwin's 1969 paper on Super Balls, arising as a second ideal collision model on equal footing with the standard specular model. A quarter century later Broomhead and Gutkin showed that–in



The trajectories with specular collisions (red) and no-slip collisions (blue) in a rectangular billiard with a scatterer.

contrast to the standard model–the trajectories of no-slip billiards are bounded on an infinite strip. Yet, three decades later the larger dynamical picture was still wide open. Our goal is to help fill this gap. Towards this end, we have:

- (With Renato Feres) Derived a very general groundwork for rigid body collisions of a large class of measurable sets in any dimension $\mathbb{R}^n$, *strict* maps obeying physical conservation laws, from which the known models in $\mathbb{R}^2$ and $\mathbb{R}^3$ emerge as special cases, and showed that the generalized billiard map preserves the natural measure, opening the door to ergodic considerations.[1]
- In my dissertation, I showed that periodic orbits are ubiquitous, including wedges with all non-escaping orbits being periodic of the same period, and the equilateral triangle with all points periodic.[2]
- (With Feres) Developed a phase portrait analysis for the higher dimensional phase spaces and showed that invariant regions are common.[3]
- (With Feres and Hongkun Zhang) Showed the existence of invariant regions precluding ergodicity in all polygons, and a larger class of linearly stable periodic orbits.[4]

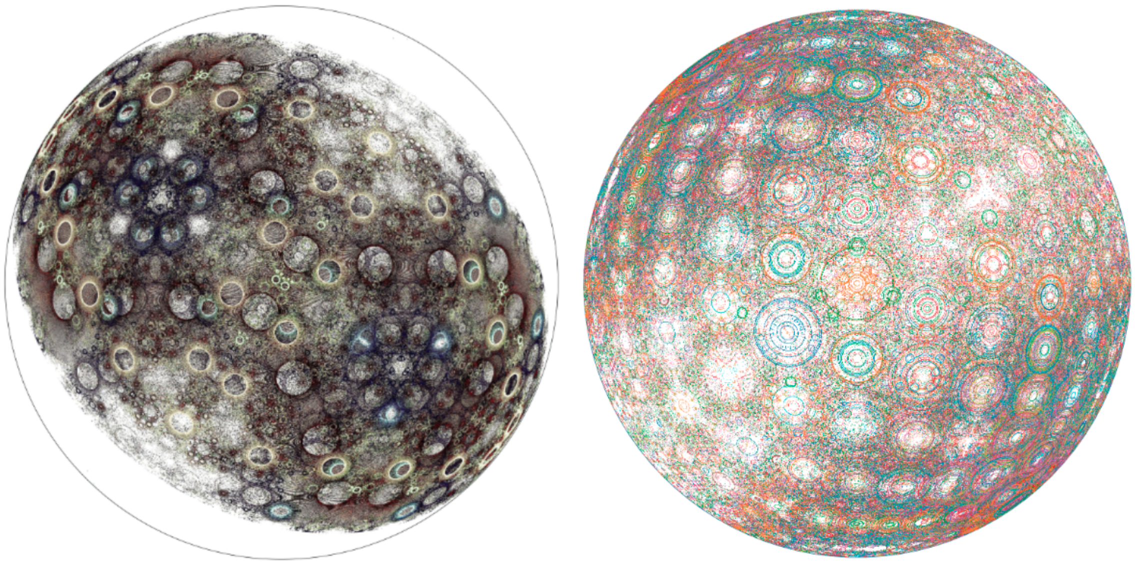
One open question: Do any ergodic no-slip billiards exist? We have shown that standard Sinai type dispersers are not ergodic in the no-slip case, but certain defocusing tables are strong candidates.

References

[1] C. Cox, R. Feres, *Differential geometry of rigid bodies collisions and non-standard billiards,* Discrete and Continuous Dynamical Systems A 36 (11), 2016, 6065-6099. [arXiv]

[2] C. Cox, "No-slip Billiards," *Arts & Sciences Electronic Theses and Dissertations*. Paper 783, (2016). [link]

[3] C. Cox, R. Feres, *No-slip billiards in dimension two*, Dynamical Systems, Ergodic Theory, and Probability: in Memory of Kolya Chernov, Contemporary Mathematics, vol. 698, Amer. Math. Soc., Providence, RI, 2017, 91-110. [arXiv]



Ubiquitous elliptic islands, as in the phase portrait of a no-slip pentagon (left) or dodecagon (right), may preclude ergodicity.

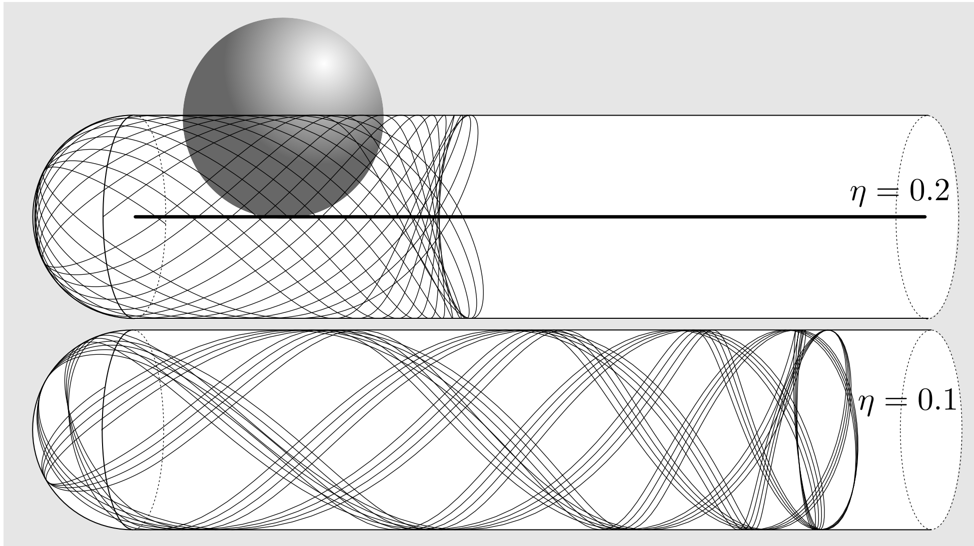
[4] C. Cox, R. Feres, H.-K. Zhang, *Stability of periodic orbits of no-slip billiards*, Nonlinearity, 31 (10), 2018, 4433-4471. [arXiv]

2.2 Nonholonomic rolling systems

Nonholonomic mechanics models systems with a certain type of constraint as exemplified by the motion of an ice skater or a ball rolling without slipping. Although the no-slip model has no explicit nonholonomic assumptions, being a rigid body model, there is reason to believe it might be a discrete analog.

- (With Tim Chumley, Scott Cook, and Renato Feres) we showed that small bounce no-slip collisions in a cylinder approximate a well-known nonholonomic rolling phenomenon.[1]
- (With Feres and Bowei Zhang) Approaching from the opposite direction of beginning with a holonomic system, Borisov, Kilin, and Mamaev showed that the small radius limit of rolling balls on the cylinder (or ellipsoid) leads to the two dimensional no-slip billiard on the strip (or circle). Building on this, we showed that in fact no-slip billiards can be derived generally as the limit of rolling systems on a manifold in any dimension.

Open question: Do rolling systems with positive radius balls demonstrate the interesting dynamics of no-slip billiards? These can be



An example of a rolling system on a line in dimension three.

modeled by numerically solving differential equations for the rolling.

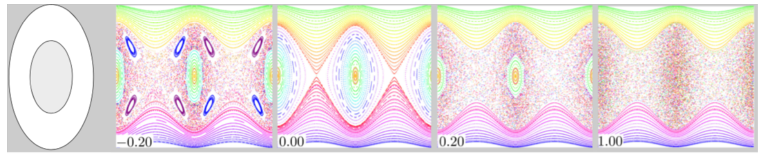
References

[1] T. Chumley, S. Cook, C. Cox, R. Feres, *Rolling and no-slip bouncing in cylinders*, Journal of Geometric Mechanics, 12 (1) 2020. [arXiv]

[2] C. Cox, R. Feres, B. Zhao, *Rolling systems and their billiard limits*, Regular and Chaotic Dynamics, 26 (1) 2021. [arXiv]

2.3 Lensed billiards

(With Chumley, Josh Covey, and Feres) An ordinary billiard dynamical system can be embedded into a one-parameter family of billiard-like systems by introducing a constant potential C supported on a subset L of the billiard domain which we call the *lens.* The original system corresponds to $C = 0$; for large enough C, the lens acts as a billiard scatterer, while for intermediate, or negative, values it acts according to the laws of geometric optics: trajectories are reflected off the boundary of L or refracted through it depending on the trajectory's angle of incidence. We call this modification of standard billiards *lensed billiard systems.* They bring to billiard dynamics some intriguing new phenomena, which we have begun to investigate. Lensed billiards share some similarities with so-called *composite* or *ray-splitting* billiards,



Changing the potential of the central ellipse can result in various dynamics, from integrable to chaotic.

although these have been studied in the context of semi-classical theory and are a greater departure from the basic theme of impulse driven classical mechanical models in the spirit of billiard dynamics. Lensed billiards are a very compelling but mostly unexplored avenue for research in billiard dynamics. It is expected that methods applied with great success to hyperbolic billiard systems will help to illuminate some of the intriguing phenomena observed numerically, particularly concerning the dependence of Lyapunov exponents on C.

One Open question: A fundamental problem is to determine how changes in potential affect known mechanisms that produce ergodicity and chaotic behavior. In addition, this research is expected to shed light on the poorly understood topic of geodesic flows on manifolds with discontinuous Riemannian metric.

References

[1] T. Chumley, J. Covey, C. Cox, R. Feres, *Chaotic lensed billiards*, in preparation.

3 Student research projects

I believe it is possible to design a project that is approachable by undergraduates while at the same time delves into relevant and interesting directions. Most of the projects have an element of numerical simulation, giving opportunities to safely attempt deeper analytic questions.

3.1 Completed projects

1. **No-slip billiards with cusps:** In a student project in a graduate course at Tarleton State University, Clayton Boone and Ed Smith looked at no-slip billiard tables with cusps. This required some advanced level coding, since

the cusps involved edges that were not linear or quadratic. The results were presented by the students at a conference and published in a conference proceedings.

C. Boone, C. Cox, E. Smith, *Specular and no-slip billiards with cusps*, Proceedings of the ICTCM, 2019. [link]

2. **First numerical search for persistently periodic polygons:** This project started in a class and was extended when I obtained a grant to fund Bishwas Ghimire for the summer. Though it did not result in a publication, Bishwas had some nice results which informed later work, and which he presented at an MAA Texas regional meeting.

3. **2019 Tarleton State University Summer Undergraduates Billiard Group:** Dr. Scott Cook and I worked with four undergraduates, including two supported by an AMS/MAA NREUP grant in support of underrepresented groups, a first generation college student supported by a Tarleton Faculty-Student grant, and a first year student also supported by an internal grant. In addition to some new results on billiards with an external force, the group discovered a very interesting generalization of a deep result by Chernov and Dolgopyat.

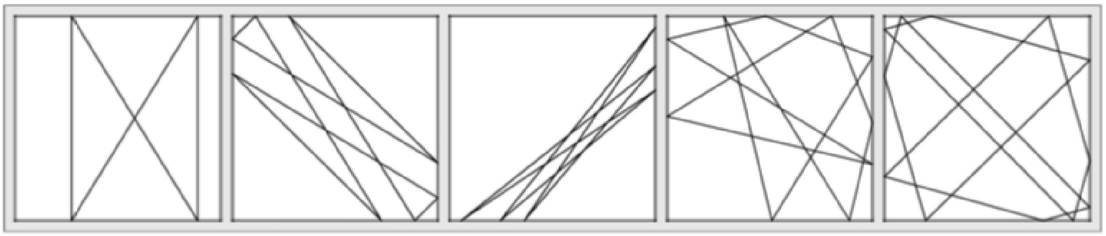
   J. Ahmed, T. Chumley, S. Cook, C. Cox, H. Grant, N. Petela, B. Rothrock, R. Xhafaj, *Dynamics of the no-slip Galton board*, submitted. (arXiv:2208.07790)

4. **University of Delaware Summer Scholars:** Over the summer of 2021 I had the opportunity to work with Jan Ahmad and Bill Wang, University of Delaware undergraduates. They immediately started producing interesting numerical results in SageMath and Python investigating no-slip billiards of variable mass distribution, showing that varying this parameter alone could determine the dynamics. Jan assisted with writing up results during the fall semester and the resulting paper was published early this year.

   J. Ahmed, C. Cox, B. Wang, *No-slip billiards with particles of variable mass distribution*, Chaos, 32 (2) 2022. (arXiv:2111.07397)

3.2 Future projects

1. **Lyapunov exponents for no-slip billiards:** Based on the work with Ahmed and Wang, several promising candidates for ergodic no-slip bil-



Numerical results suggest that certain mass distribution particles result in persistent periodicity.

liards have been identified. Algorithms are known for calculating Lyapunov exponents and could be implemented quickly by an interested undergraduate to test the candidates. (A grant from Mount Holyoke College for undergraduate research is tentatively allotted to pursuing this project.)

2. **Proving persistently periodic polygons:** One of the most interesting phenomenon of no-slip billiards is the apparent existence of persistently periodic billiards, in which phase space is comprised entirely of periodic orbits. New examples were uncovered numerically in the Ahmed/Wang project, but none have been proven to be persistently periodic analytically. An undergraduate interested in dynamics and ready to take on a combinatorics analysis might be able to provide these proofs.

3. **Rolling systems of polyhedra:** With the connection between rolling systems and no-slip billiards now established, nonholonomic rolling systems are of greater interest. The underlying math needed to experiment with rolling on polyhedra is known, but to date no work has been done. It would be especially interesting to see which dynamic properties of no-slip billiards can be reproduced in rolling systems.

4. **Machine learning of billiard dynamical systems:** Zhang, Chumley, and I have been able to establish the feasibility of using neural networks to learn billiard dynamical systems, but to date we have only investigated a few tables representing a small fraction of the variety of dynamics present in standard and alternative billiard models. A student with strong coding skills would be able to immediately make meaningful contributions in this investigation.